# Frequency Response and Filters

## Intro

### ILOs

1. Be able to determine the frequency response (magnitude and phase) of a circuit containing any combination of R, L, and C elements with a defined input & output.
2. Identify low-pass, high-pass, bandpass, and band-stop filters, whether in a simple or reasonably complex form
3. Determine cutoff frequency and bandwidth & quality factors (where defined) of filters
4. Produce and read Bode plots and calculate gain in dB.
5. Design simple RC, RL, and parallel/series RLC single-stage filters to meet gain & bandwidth specifications.

Diagram

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### Topic 4 Videos:

1. Intro: <https://www.youtube.com/watch?v=AUWbgpT6qoU&list=PLhbHWgMknRJT_eKLFXB843NkaNHfJ37Pw&index=15>
2. Extra Problem Videos:
   1. <https://www.youtube.com/watch?v=2smSVnLmMeE&list=PLhbHWgMknRJT_eKLFXB843NkaNHfJ37Pw&index=16>
   2. <https://www.youtube.com/watch?v=XLKG57cZnbY&list=PLhbHWgMknRJT_eKLFXB843NkaNHfJ37Pw&index=17>
   3. <https://www.youtube.com/watch?v=9rTc9TWa0bY&list=PLhbHWgMknRJT_eKLFXB843NkaNHfJ37Pw&index=18>
   4. <https://www.youtube.com/watch?v=9O6r452AGxE&list=PLhbHWgMknRJT_eKLFXB843NkaNHfJ37Pw&index=19>
3. Lab Skills: <https://www.youtube.com/watch?v=4FpGG_gmzLQ&list=PLhbHWgMknRJT_eKLFXB843NkaNHfJ37Pw&index=20>
4. Hantek Lab Modification: N/A (no new info needed for doing the lab with the Hantek this week)

### Deliverables

Note: As always, your full objective for this topic is to review these notes, the videos, practice problems, live class sessions and forum content, then to write-up creating and tri-solving a variation problem of the week's topic that demonstrates you've mastered the content. With that all still in mind, following are some specific guidelines & tips for this week.

See the Outline and Deliverable Rubric files for information for the write-ups in general. Specific tasks for your H4 deliverable are as follows:

1. Build a circuit similar to the ones tackled below in the example problems which has an input AC voltage source with one side at ground and a clearly-defined output node relative to ground. Incorporate an interesting combination of R, L, and C elements so you can demonstrate your mastery of the topic.
   1. Make sure that your power supply's current stays below 100 mA at all frequencies of interest.
2. Analytical: calculate the transfer function, create a Bode plot describing the circuit, determine which type of filter it is, and calculate the "centre frequency" (or the frequency with the most or least gain, as appropriate for your filter), and the -3dB frequency (or frequencies) in Hz.
   1. Note: Typically you should define the -3dB frequency relative to the "pass band"; i.e., if your circuit is a low-pass that has a max transfer function of 0.5 in the pass band -3dB should be where the amplitude reduces to 1/sqrt(2) of this rather than 1/sqrt(2) of the input (since that never happens). However, for some filter circuits it makes more sense to define the -3dB frequency relative to the resonance peak, especially if you have a -3dB compared to that peak on both sides of it. You should use whichever definition makes more sense for your circuit, as long as you explain your reasoning and are clear about what the -3dB is relative to.
3. Simulate your circuit in Multisim and create a bode plot. Use it to measure the same frequencies (centre and -3dB) you found analytically, as appropriate for your circuit.
4. Physically build the circuit and measure the output amplitude gain and phase shift (relative to the input) at several frequencies to obtain the datapoints necessary to create a bode plot and compare with your analytical and simulation work. It will be time consuming to take the measures to produce a detailed plot, so choose only 8-10 points that will give you a good idea of the important features of your frequency response curve.
5. Analysis: as usual

Note: Unfortunately, the Hantek doesn't let you see the output while you're moving the frequency, so be sure to watch the physical lab video to get a better visualization of what's going on: <https://www.youtube.com/watch?v=4FpGG_gmzLQ&list=PLhbHWgMknRJT_eKLFXB843NkaNHfJ37Pw&index=20>

## Frequency Response

### Example

Consider the following circuit:



Previously (in Topic 2) we learned how to determine the voltage vs. time at the output node:

**> restart:**

**f:=10e3: omega:=2\*3.14159\*f:**

**C:=1e-6: L:=.1: R:=100:**

**ZC:=1/(I\*omega\*C); ZL:=I\*omega\*L;**

**Vi:=1:**

**ZRight:=ZC+ZL;**

**ZTop:=R/2 + 1/(1/ZC+1/ZL);**

**Vo:=(ZRight/(ZRight+ZTop))\*Vi; convert(%, polar);**













Meaning that .

In a way, the passive network (i.e., this particular arrangement of resistors, capacitors, and inductors, and placement of the input supply and output nodes) is a *system* which transforms the *input* voltage signal provided by the supply into an *output* voltage signal we can measure or use to power other things:

Diagram, schematic

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Diagram, schematic

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Looking at the solution for output voltage compared to input voltage in the phasor domain, we had that . Due to the *linear* nature of the system (i.e., the fact that it is made up of only inductors, capacitors, and resistors, all of which have linear differential equations as current-voltage relations), the output voltage was directly proportional to the input supply voltage (e.g., if we doubled the input voltage V1 the output voltage would also double).

The proportionality constant between the input and output voltage is called the **gain** (**G**)of this system(technically this is the **voltage gain**; unfortunately, even within the context of electronics alone, "gain" is an ambiguous term: <https://en.wikipedia.org/wiki/Gain_(electronics)>);



(Note the use of  vs. ; [voltage] gain *is defined as* the ratio of the output to input voltage phasor, which in this case (but not necessarily always) happens to be equal to .)

Notice that the impedances (and therefore the gain) depend on the *frequency* the input voltage sinusoid; so this "system gain" is not actually just a property of the system itself, but also of the input signal's frequency. We can capture this idea by instead leaving the input frequency as a variable and computing the frequency-dependent gain function:

**> restart:**

**#f:=10e3: omega:=2\*Pi\*f: #commented out; no frequency assumed!**

**C:=10^(-6): L:=1/10: R:=100:**

**ZC:=1/(I\*omega\*C); ZL:=I\*omega\*L;**

**#Vi:=1:**

**ZRight:=ZC+ZL:**

**ZTop:=R/2 + 1/(1/ZC+1/ZL):**

**Vo:=(ZRight/(ZRight+ZTop))\*Vi:**

**H:=Vo/Vi;**







***This*** frequency-dependent gain function of the system ***is***a property of the system alone (independent of the input signal) and is often called the system's **transfer function,** . Transfer function is the ratio of the system's output to input (and the input and output could be defined as voltage levels like they were here, as current, or even one of each).

To explore how this gain behaves, we can make a plot of its magnitude and phase vs. frequency:

**> plot([abs(H), argument(H)], omega=1e2..1e5);**

A picture containing graphical user interface

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However, because the transfer function is a rational function in , it's typically easier to see how it's behaving by making a log plot its magnitude and a linear plot of its phase both vs. the log of the input frequency, as follows:

**> with(plots):**

**omega\_low:=1e2: omega\_high:=1e5:**

**dualaxisplot(**

**loglogplot(abs(H), omega=omega\_low..omega\_high, legend='magnitude'),**

**semilogplot(argument(H)\*180/Pi, omega=omega\_low..omega\_high, color='blue', legend='phase'),**

**gridlines=true);**





Chart

Description automatically generated

From this plot we can see that something special happens at and around ; it turns out this is where the inductor and capacitor perfectly cancel out, leading to no output at all. But just below this frequency and just above this frequency the system gives larger than unity gain in amplitude, but large phase shifts; clearly some pretty wild behaviour to be aware of and explore here!

This unit is on examining the frequency response of systems, i.e., better understanding passive networks by exploring how, as a function of frequency, they'll transform input signals into output signals. This will not only allow us to understand AC network analysis at a much deeper level, but also understand and design basic filters.

*Q:* What is the transfer function a function *of*?

*A:* Input signal [angular] frequency.

### Frequency Response

The transfer function of a system (the ratio of the output to the input of a system as a function of frequency) is often called the system's **frequency response** because it captures how it responds to all possible input frequency sinusoids. Depending on whether the output and input are both voltage, both current, or different we may use the symbol **G** (gain) or **H** (impedance or admittance, but often "transfer function" in general, including gain) for frequency response:

For a system than turns input voltage phasor  into output voltage phasor  the frequency response (transfer function) is the voltage gain .

Similarly, the frequency response for the other three system input/output combinations are:

|  |  |  |
| --- | --- | --- |
|  | Input | Input |
| Output | Voltage gain | Impedance |
| Output | Admittance | Current gain |

To find the frequency response of a circuit, you do the same AC analysis as you did in chapter 4 (except leaving the frequency as a variable rather than substituting its numeric value) and write the output in terms of the input.

After you write the frequency response and factor it into separate linear and quadratic terms in the numerator & denominator, you may have something like this:



Roots that appear in the numerator are called **zeroes** (the frequency response is 0 at those frequencies), while those that appear in the denominator are called **poles** (the frequency response magnitude goes to infinity at those frequencies).

A linear term (here,  or  or ) is called a **simple** pole or zero, while a quadratic term (here ) is a **quadratic** (or **complex**) pole or zero.

## Filters

Low- and High-Pass Filters

Notice that the frequency response of capacitors and inductors are inverses:

1. A capacitor limits to zero impedance at high frequency and infinite impedance at DC (low frequency): 
2. An inductor limits to infinite impedance at high frequency and zero impedance at DC: 

### Low-Pass Filters

Using information about capacitor and inductor behaviour at frequency extremes, we can create **low-pass filters:** circuits which allow only *low* frequencies on the input to pass to the output and stop *high* frequencies from passing. e.g.,

Diagram, schematic

Description automatically generated

Here, , so the transfer function is 

If  we have  - the inductor becomes a short, so the input is just passed through completely.

On the other hand, as  we have ; the output goes to zero (while the phase limits to -90°). Therefore, this is a **low-pass** **filter** - it totally passes (i.e., gain → 1) low enough frequencies while totally stopping (i.e., gain→ 0) high enough frequencies.

(specifically, this is an "RL lowpass filter" because it's made with a resistor and inductor).

 → so this has a "high frequency rolloff" with .

You can also make a low-pass filter with a resistor and capacitor (an "RC lowpass filter"):

Chart, diagram, schematic, box and whisker chart

Description automatically generated

Here, , so we have similar behaviour to the RL filter but with RC in place of L/R.

(déjà vu - RC and L/R were the timeconstants of first-order circuits, which isn't so surprising since these are the same circuits and those combinations are needed so the units work out when multiplying by frequency).

You can also combine these effects into more complicated versions of these filters. e.g.,

Diagram, schematic

Description automatically generated

At low frequencies, the capacitors limit to open circuits while the inductor limits to a short, so  (at DC), while at high frequency the opposite happens and .

You can *calculate* the exact transfer function and directly confirm this intuition as well:

**> restart;**

**ZL:=I\*omega\*L: ZC1:=1/(I\*omega\*C1): ZC2:=1/(I\*omega\*C2):**

**Zp2:=1/(1/ZC2+1/Ro): Zp1:=1/(1/ZC1+1/(ZL+Zp2)):**

**Va:=Vi\*Zp1/(Rs+Zp1):**

**Vo:=simplify(Va\*Zp2/(ZL+Zp2));**



Indeed, when  this limits to , while when  it limits to 0.

Therefore, this still stops high frequencies like the simple RC & RL lowpass filters, but only *partially* passes low ones (it attenuates them by ). You could still call this a low-pass filter, but technically it's more like an attenuator combined with a lowpass filter (a true low-pass filter should pass the low frequencies without attenuating them). Sometimes you might build a lowpass filter that doesn't totally cutoff the high frequencies either, but just attenuates them.

e.g., this filter:

Diagram, schematic

Description automatically generated

At low enough frequencies, the capacitor is an open circuit so this filter has an output gain of 50% (i.e., ).

But at high enough frequencies the capacitor becomes a short in which case R2 and R4 in parallel combine so that less voltage shows up across them, only 33% of the input: . You might consider this a low-pass filter as well, but more specifically it's a 50% attenuator combined with a low-pass filter that attenuates high frequencies by a *further* 33% (i.e., overall gain of 50%, then low-pass filter that has gain of 100% for low frequencies and 67% for high frequencies. Gain of 67% means 33% attenuation).

*OK, sure, these filters pass low frequencies and limit high ones, but where's the cutoff between low and high?*

The **cutoff frequency** of a low-pass filter is defined as the frequency at which only half of the input power is passed to the output.

Note that because power scales like voltage squared, it's fair to say that at the cutoff frequency ; i.e., at the cutoff frequency the magnitude of the [voltage gain] transfer function has dropped to 

Note that if the filter also has some attenuation too, we'd first need to factor this out and define the cutoff frequency as where the voltage has dropped to  of whatever it is at low frequencies. That is, .

#### Example - Cutoff Frequency

Find the cutoff frequency of the following circuit and make a log-log plot of the transfer function magnitude vs. frequency and a lin-log plot of its phase vs. frequency.

Diagram, schematic

Description automatically generated

Solution:

The transfer function is 

The cutoff frequency is the frequency where . The max gain () is 100% (this filter is not attenuating), so we're looking for  such that 

Substituting into the transfer function:



In this case, that's , so that .

We can confirm this and plot magnitude and phase vs. frequency (i.e., the **frequency response** of the circuit) as follows:

**> restart:**

**L:=.1: R:=100:**

**H:=R/(R+I\*2\*Pi\*f\*L):**

**with(plots):**

**Hmag:=abs(H);**

**fco:=fsolve(Hmag=1/sqrt(2), f=100..800);**

**HphaseDeg:=argument(H)\*180/Pi;**

**loglogplot(Hmag, f=10..1e5, Magnitude=1e-3..1);**

**semilogplot(HphaseDeg, f=10..1e5, Phase=-90..90);**







Chart, line chart

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These plots are also available in Multisim using the AC Sweep analysis:

A picture containing diagram

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*Deep Thoughts: The cutoff frequency was the inverse of that L/R factor, why?*

Because the transfer function was , and this has a magnitude of  when .

*Does that mean for an RC low-pass filter it will be 1/RC?*

Yes.

*Neato.*

### High-Pass Filters

If we swap the inductors and capacitors in any of the previous circuits we'll arrive instead at circuits that pass high frequency and stop low frequency signals: **high-pass filters**. For example, this is an RC high-pass filter:

Diagram, schematic

Description automatically generated

while this is an RL high-pass filter:

Chart, diagram, schematic

Description automatically generated

Here's a more complicated partly-attenuating multi-stage high-pass filter:

Diagram, schematic, box and whisker chart

Description automatically generated

As before, you can analyze this by inspection by noting that at low frequency the inductors short out the output while the capacitor acts as an open circuit blocking it off, and at high frequency the inductors are open while the capacitor is a short to get a transfer function of  as , and you can also find the full frequency-dependent transfer function to confirm it:

**> restart;**

**ZC:=1/(I\*omega\*C): ZL1:=I\*omega\*L1: ZL2:=I\*omega\*L2:**

**Zp2:=1/(1/ZL2+1/Ro): Zp1:=1/(1/ZL1+1/(ZC+Zp2)):**

**Va:=Vi\*Zp1/(Rs+Zp1):**

**Vo:=simplify(Va\*Zp2/(ZC+Zp2));**



**Cutoff Frequency Summary:**

Simple high- and low-pass RC & RL filters limit to passing 100% and stopping 100% of signals as frequency limits to its extremes, and partially pass & stop frequencies between these extremes. The **cutoff frequency** (AKA **half-power frequency**, AKA **-3dB frequency\***, AKA **corner frequency**, AKA **break frequency**)\*\* of a filter is the frequency where the power gain is half of the maximum power gain.

(i.e., if a filter has a max amplitude gain of Gp at extremely low or high frequency, then the cutoff frequency would be where the amplitude gain is Gp/sqrt(2)).

e.g., for an RL lowpass filter, , with a magnitude of . The max gain magnitude is 1 (at DC), and therefore the cutoff frequency is the frequency where . Specifically, .

Notice also that the *phase shift* of the output relative to the input at the cutoff frequency is -45o:



\*more on "-3dB frequency" in the "Decibels and Bode Plots" section, 4.2.4.

### Bandpass Filters, Resonance, and Quality Factor

For an RLC series circuit with output across the R,

Diagram, schematic

Description automatically generated

we have the frequency response



We can plot this using Maple, or using an AC sweep in multisim:

A screenshot of a computer

Description automatically generated with low confidence

Notice the large flat region in the magnitude plot where the gain is 1 (the "passband"). This combines the properties of a lowpass and highpass filter together: at very low frequencies the capacitor impedance gets very large and sends the gain to zero while the inductor does the same at very high frequencies. However, there's a large region in the middle where both inductor and capacitor impedance are negligible compared to the resistor so the voltage divider makes the transfer function a 1:

A screenshot of a computer

Description automatically generated with medium confidence

More technically, we can define the **passband** (i.e., **passband bandwidth**) of the filter as the frequency range between the cutoff frequency on either side; that is, the passband is the range of frequencies between each frequency where  (or more generally where  if the system is a filter combined with an attenuator and also attenuates in the passband).

Note that in this filter the signal *does* get attenuated at the edges of the passband, and doesn't get *completely* cutoff until well past the cutoff frequency. i.e., the low- and high-frequency ***roll-off****s* are not very sharp. This is because in those ranges signal only drops off proportional to  - i.e., this is only a second-order bandpass filter with 2 energy storage elements, leaving only one responsible for each roll-off. You can improve this sharpness by building more complicated higher-order filters.

For a wide bandwidth filter like this one, it's a good approximation to ignore the inductor impedance at the low frequency end, and say that 

and similarly ignore the capacitive impedance at the high-frequency end to find:  


making the bandwidth the range, or in cyclical frequency, .

Note that for the filter to behave like this and actually *have* a passband the resistance needs to be high enough that there's a region where it can overpower the impedance of the inductor and capacitor. If the resistance is too low then there's no longer a large flat frequency range where the resistor dominates the voltage divider, and instead this filter is mostly zero but with a spike where the gain is still 1:

**> restart:**

**C:=1e-6: L:=.05: w:=2\*3.14159\*f;**

**ZC:=1/(I\*w\*C): ZL:=I\*w\*L:**

**Habs:=abs(R/(R+ZC+ZL));**

**Rvals:=seq(10^n, n=1..7);**

**Hvals:=seq(subs(R=Rvals[n], Habs), n=1..7):**

**with(plots):**

**semilogplot([Hvals], f =10^(-3)..10^9);**







A picture containing text, kitchenware

Description automatically generated

*But wait! WHY is there a spike??!*

The spike means that no matter now low R is, there's *some* frequency (which seems independent of R) where the gain is still 100% → all of the signal appears across the resistor. This is because the impedance of the inductor is always a positive imaginary number *directly* proportional to frequency  while that of the capacitor is always a negative imaginary number *inversely* proportional to frequency , and therefore no matter what size L and C are, there's going to be *some* frequency where  and 100% of the signal shows up across the resistor (no matter how small *it* is).

*Tell me more about this magical frequency‼!*

This frequency is called the ***resonant frequency*** of the circuit (also called the **centre frequency**, since it's in the centre of the passband on a log scale). It's located at



At this frequency, the inductor and capacitor in series have *cancelled* to give zero impedance, maximizing the current through this circuit because the supply sees only the resistor: . However, even though the series combination of impedance is zero, *individually* they still have equal-and-opposite *nonzero* impedances  and , and so would have voltages across them of  and .

These voltages are also equal and opposite so add to zero (which they must from Kirchoff's law since all the input voltage is across the resistor), but they're not zero themselves. In fact, depending on the parameter values,  could be very large, much larger than 1.

#### Example - Resonance

Find the resonant frequency and resonant voltage amplitude across the inductor in the following circuit:

Diagram, schematic

Description automatically generated

Solution:

Inductor and capacitor cancel when 

At this point the input voltage is all placed across the resistor, , so that , specifically,

**> restart:**

**C:=1e-9: L:=.1: R:=100: Vin:=1:**

**wres:=1/sqrt(L\*C);**

**fres:=wres/(2\*3.14159);**

**ZL:=I\*wres\*L;**

**VLres:=ZL/R\*Vin;**

**VLresCheck:=I\*sqrt(L/C)/R\*Vin;**











That is, our measly 1 V input has (at just the right frequency) caused a 100 V amplitude across the inductor!

*No way! I gotta see this!*

Way. Check it out; you can rearrange the circuit so the output is across the inductor instead, which would make it behave like a high-pass filter, except that at resonance it acts like a 100x amplifier!

A picture containing diagram

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*But WHY??? How!??*

What's going on is that energy is being tossed back and forth between the inductor and capacitor. Like a mass on a spring, when you push at *just* the right frequency the energy you add each cycle can build up over time so that after enough cycles its amplitude is much larger than you'd expect from the force you're applying and the spring by itself.

*OK but it's not really like a mass on a spring, it's a circuit… right?*

In fact, Newton's 2nd law for a mass-spring-damper with applied force  says



While here in our RLC circuit Kirchoff's voltage law says



Taking a derivative with respect to time, we have:



And the so the two systems have totally analogous DEs, where

1. the instantaneous current is like the instantaneous position of the mass
2. the applied voltage's time derivative (which is also a sinusoidal signal) is like the driving force
3. the inductance is like the mass
4. the resistance is like the damping ratio
5. the inverse of the capacitance is like the spring constant (*low* capacitance makes the system very tough to build up high current in; just like how a stiff spring makes it very tough for the mass system to build up high amplitude)

*Whoa.*

Yeah!

#### Resonance Advanced Terms - Quality Factor and Damping Ratio

Returning to the RLC series circuit with the output across the resistor:

Diagram, schematic

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The transfer function, after rearranging to eliminate fractions in fractions, is:



Since we know  is a pretty special frequency for this circuit, we can try to rewrite the transfer function using this instead of *L*. When we do that, it turns out we can also eliminate R and C together if we also define the **damping ratio** as  (analogous to mass-spring systems where ), so that  and



Therefore, L & C determine the centre frequency by themselves, and then the size of R compared to the impedance of L & C there determines the width of the passband (and whether it exists at all instead of a spikey resonance effect).

In some systems we actually want that spikey resonance effect (an example is positive feedback to help zero in on specific frequency for timing circuits, or if we're trying to make a measurement at a very precise frequency and reject all other others). In those cases where damping ratio gets very small it's more convenient to talk in terms of the **quality factor**: . For more complex resonant systems which aren't just RLC series circuits, quality factor is defined the ratio of the resonant frequency to the bandwidth: ; which is consistent with the definition here (as long as *Q* > about 3). High quality factor represents a narrow pass-band and sharp filtering.

### Bandstop Filters (AKA Notch Filters)

If we instead take the output across the inductor & capacitor in series,

we have the opposite effect: a filter that passes all the frequencies *except* those in a band around resonance. This kind of filter is called a **band stop filter** or **notch** **filter**:

In general (i.e., for circuits other than series or parallel RLC), the resonant frequency of a circuit with inductors and capacitors is the point where the inductive and capacitive reactances combine to an extrema in magnitude; This could be a maximum or minimum in the gain (or even neither of these), depending on where the output is taken relative to the input.

### Summary & Tips

Tips for filter analysis, part 2:

1. An inductor in **series** with a capacitor
   1. Diagram

      Description automatically generated
   2. has zero impedance at resonance:; zero when , limits to infinite as  goes to 0 or infinite
2. An inductor in **parallel** with a capacitor:
   1. Diagram, schematic

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   2. has infinite impedance at resonance: ; infinite when , limits to zero as goes to 0 or infinite.

If a renegade resistor shows up messing up any of these perfect resonance effects then the resonance effect might still be pretty prominent as long as the resistor impedance is ignorable compared to the capacitor or inductor impedance at resonance.

e.g.,

Diagram, schematic, calendar

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Resonance (ignoring the resistor) happens at  where . This means the resistor only amounts to a 1% impedance difference next to the inductor at the resonance frequency, so we can safely ignore the resistor and use this to build filter like it's not even there, e.g., here's a narrow-notch filter using this:

Diagram, schematic

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A screenshot of a computer

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Which is almost the same as the response it would have without the resistor at all:

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Remember that if for some reason you have the resistance *in parallel* with a component it will only be ignorable if it's a much *larger* impedance; this resistor is completely messing up the resonance effect:

Diagram, schematic

Description automatically generated

Again  at  rad/s, meaning that this time it's the *capacitor* that's not doing anything; the resistor is "shorting" its resistance to 100 Ω from well *above* resonance and any lower frequency, and the capacitor won't start to short out the resistor until above 1e7 rad/s, but the series inductor is already 100x the impedance by 1e5 rad/s, and is 10000x it by 1e7, so the circuit will behave the same if the capacitor isn't there at all.

### Decibels and Bode Plots

A **Bode plot** is a pair of plots of the magnitude and phase of a filter's frequency response; the magnitude plot is a log-log plot while the phase plot is lin-log. The plot earlier was almost a bode plot, but typically the magnitude of the gain would be expressed in terms of **decibels**, where .

*e.g., a "gain of 1" corresponds to 0dB, while a gain of 0.5 corresponds to -6dB.*

Note that the "**cutoff"** frequencies (with ) correspond to -3dB, and for this reason the cutoff frequencies are also called the "minus three dee-bee frequencies" or just "three dee-bee points" for short.

Aside: 1 bel corresponds to a power gain of 10, and the number of bel of gain is the log10 of the gain in the *power*. Since power is proportional to voltage amplitude squared, the gain in power is , so the number of bel [B] of gain is . A decibel [dB] is 0.1 bel, and therefore .

This is the reason that a 3dB higher sound level corresponds to double actual sound energy whereas here 3dB higher corresponds to only sqrt(2) higher amplitude.

Concluding note: single-stage RC, RL, and RLC filters are simple to analyze but limited in their usefulness; most real applications demand filters with sharper cutoffs than the kind realized by the filters in this chapter. One improvement is stacking multiple stages to achieve quadratic or better roll-offs. Further improvements come with **active** filtering (using active rather than passive circuit elements; more on this in your third year circuits course).

### An Example Problem (everything covered)

Below is taken from a sample lab in a previous year:

Diagram

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**Figure 1**

**Step 1:** For the circuit in Figure 1, calculate the transfer function, determine which type of filter it is, then calculate the centre frequency and -3dB frequency (or frequencies) in Hz.







maximum gain is at  (resonance; AKA the centre frequency)



The transfer function is

**restart;**

**ZL:=I\*omega\*L: ZC:=1/(I\*omega\*C):**

**Zpar:=1/(1/ZL+1/ZC+1/R):**

**H:=simplify(Zpar/(R+Zpar));**

**L:=.1: C:=1e-6: R:=100: H;**





giving , so that .

With C & L in parallel, resonance occurs when ZC & ZL combine to infinite impedance. Since that's in parallel with the output resistor, this is a **bandpass** filter.

The resonance (AKA "centre" frequency) is



The -3dB frequencies are the points on either side of the resonance where . In this case, , and will reduce to this over sqrt(2) when , meaning

**> solve(5e-5\*omega-500/omega=1);**

**solve(5e-5\*omega-500/omega=-1);**





so +20488 and 488 rad/s, or 3261 Hz & 77.7 Hz. The bandwidth is the separation between these: B = 3261-77.7 = 3183.1015 Hz.

At high quality factor, the quality factor is the ratio of the resonant frequency to the bandwidth. In this case, we'd find Q = 0.158, which is too low for this definition of Q to really be applicable.

**> with(plots):**

**loglogplot([abs(H), 0.5/sqrt(2)], omega=100..1e5, numpoints=1000, gridlines=true);**

Chart, line chart

Description automatically generated

(alternative plot syntax:

**> restart;**

**ZL:=I\*omega\*L: ZC:=1/(I\*omega\*C):**

**Zpar:=1/(1/ZL+1/ZC+1/R):**

**H:=simplify(Zpar/(R+Zpar));**

**L:=.1: C:=1e-6: R:=100: H;**

**with(plots):**

**loglogplot([abs(H), 0.5/sqrt(2)], omega=100..1e5, numpoints=1000, gridlines=true,   
labels=[typeset("Angular Frequency ",omega," [rad/s]"), typeset("Voltage Gain ", [unitless])],   
legend=[abs('H'), (1/2)/sqrt(2)],   
labeldirections=[horizontal,vertical]);**





Chart, line chart

Description automatically generated

)

Note: the log-log plot is not in dB; you can get one in dB directly by plotting 20log10(abs(H)) as a lin-log plot ("semilog" in Maple):

**> with(plots):**

**semilogplot([20\*log10(abs(H))], omega=100..1e5, numpoints=1000, gridlines=true);**

Chart, line chart

Description automatically generated

(alternative syntax:

**> semilogplot([20\*log10(abs(H)), 20\*log10(0.5/sqrt(2))], omega=100..1e5, numpoints=1000, gridlines=true,**

**labels=[typeset("Angular Frequency ",omega," [rad/s]"), typeset("Gain [dB]")], legend=[abs('H'), (1/2)/sqrt(2)],**

**labeldirections=[horizontal,vertical]);**

Chart, line chart

Description automatically generated

)

**> semilogplot([argument(H)], omega=100..1e5, numpoints=1000, gridlines=true);**

Chart, line chart

Description automatically generated

**Step 2:** Simulate the circuit of Figure 1 in Multisim and determine the centre frequency, gain at the centre frequency, and the -3dB frequencies. Compare them to your results from step 1.

Can use either AC sweep or the Bode Plotter tool. AC sweep is likely easier to read; it's OK that it's not in decibels.

A picture containing line chart

Description automatically generated

**Step 3:** Build the circuit in Figure 1 on a breadboard and use the oscilloscope to measure the centre frequency, gain at the centre frequency, and the -3dB frequencies of the filter. How well do these results agree with your simulations, accounting for experimental uncertainty?

**Step 4:** Vary the frequency of the input signal over a wide range and confirm for yourself that the bandpass filter is only allowing a small band of frequencies to pass. At some frequency off resonance, look at the output and input signals on the oscilloscope at the same time, is there a phase difference between the two? How does this phase difference vary with frequency?

#### Followup question: Flatness of the bandpass

Without doing any algebra or simulations, predict whether increasing or decreasing R2 should cause the passband to get flatter & wider, and try to repeat for R1. Test your theory.

Diagram

Description automatically generated

Answer:

A flatter passband will happen when the resistance stops the capacitor and inductor from mattering at the same time. For this circuit the LC components are in parallel, so the one that matters most in a given frequency range is the one with *lower* impedance, and we can reduce the range where they both matter at once by making R1 a *low* impedance compared to their impedance magnitude at resonance.

i.e., making R1 *lower* will widen the resistor-dominated region between the capacitor-dominated high frequency rolloff and the inductor dominated low-frequency rolloff, while making R1 higher will sharpen the peak.

R2's affect is a bit trickier, but keeping in mind that the complex impedance that dominates is the lower magnitude one when the capacitor and inductor are in parallel, we can similarly predict that *reducing* R2 should also make it harder for the inductor or capacitor to matter at similar times, this time from a voltage divider effect rather than actually shorting them out like R1 was, so is not as strong of an effect as it was.

*Try it out in multisim and see!*

#### Further Analytical Results: Resistance of inductor?

The inductors in the lab (like all non-superconductor inductors) aren't perfect, so we can't actually build the circuit above. How does the transfer function change if we include a  resistor in series with the inductor?

Diagram, schematic

Description automatically generated

Note that the original resonance was  where . This is not so large that the 200 Ω resistor is negligible compared to the inductor, so this ought to change the circuit response by a fair bit around resonance.

Furthermore, the ability of this filter circuit to achieve a low frequency rolloff requires that the inductor can become a negligible impedance compared to R1, but this resistor means that instead, the low frequency limit only brings the parallel branch impedance down to , i.e., , so the voltage divider between that and R2 doesn't go down to 0 but only 67/167 = 40%. It's hard to even consider this a bandpass filter at this point rather than a lowpass one:

A picture containing graphical user interface

Description automatically generated

### Frequency Response Extension via Fourier Analysis

In this section we've seen how to understand a system's frequency-dependent response to input sinusoidal signals via the transfer function. Because these systems are linear, this means we could also determine the response to multiple sinusoids at once. e.g., suppose the input signal was the sum of two different supplies at different frequencies:



If we know the transfer function of this system, then because the system is linear we could still determine the output via superposition; specifically:



This linearity aspect is extremely useful when combined with Fourier analysis (writing a periodic signal as the sum of its Fourier series, a linear combination of sinusoids): Since the frequency response shows how the system responds to all possible input sinusoids and a function's Fourier series shows how to express any periodic signal as a linear combination of sinusoids, because the system is linear we can use these pieces of information to determine the response to any input signal (e.g., not just sine waves, but also square waves, sawtooth waves, step functions, etc.)

### Tip: Maximize in Maple

Especially when incorporating the real resistance in series with a physical inductor, you can end up with complicated functions to maximize analytically if you want to find where the maximum gain occurs. e.g.,

A picture containing chart

Description automatically generated

*Difficult way #1:*

In these cases, you may try to simplify the denominator so you can take the absolute value and look for where it's maximized by looking for critical points (e.g., set the derivative to 0 or undefined); which you could do analytically after significant effort:



etc.

*Difficult way #2:*

But you might use Maple to help with this too.

Maple can't simplify abs(H) because it doesn't know whether omega is real or not:

**> restart:**

**H := 1/(0.0012084\*I\*omega + 1.604804805 + 1208.4/(220.1 + 0.1\*I\*omega));**

**simplify(abs(H));**





You can fix this by telling Maple to assume omega is positive, like this:

**> restart:**

**assume(omega, positive):**

**H := 1/(0.0012084\*I\*omega + 1.604804805 + 1208.4/(220.1 + 0.1\*I\*omega));**

**simplify(abs(H));**





**> restart:**

**assume(omega, positive):**

**H := 1/(0.0012084\*I\*omega + 1.604804805 + 1208.4/(220.1 + 0.1\*I\*omega));**

**simplify(abs(H));**





Then you can manually take the derivative, set it to zero to find critical points, and look for maximum values.

*Easy way:*

But an even better way is to plot the function and use maximize; i.e.,

**> restart:**

**H := 1/(0.0012084\*I\*omega + 1.604804805 + 1208.4/(220.1 + 0.1\*I\*omega));**

**with(plots):**

**loglogplot(abs(H), omega=10..10e5);**

**maximize(abs(H), omega=10^3..10^4, location);**



Chart, line chart

Description automatically generated



The command " **maximize(abs(H), omega=10^3..10^4, location);**" tells Maple to numerically find the highest value of abs(H) in the given range of omega (**10^3..10^4**), and the **location** keyword tells it to then let you know not just what the max value was (0.2777…) but the corresponding value of omega where it occurred (3336.9…).